Galois-Teichmüller theory
Arithmetic geometry principles

Grothendieck, a Multifarious Giant
Mathematics, Logic and Philosophy

Chapman University – 28 May, 2022

Grothendieck Pillar

Discrete [NT] vs **Continous** [Geo]

- **Category:** ``shapes", skeleton
- **Topology:** fibre, b. change
- **Encounter:** Thurston, Thompson Moduli, Tate-Faltings,...

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Étale Fundamental Group – From SGA1 to Esquisse

Let X be a space over a field $\mathbb{Q}$ (e.g. scheme, DM stack), $*$ geo. pt.

Galois category (Ét. covers of X, fibre Funct. F_*) **[Category]**

$$\pi_1^{et}(X,*) = \lim_{Fet_X} Aut(F_x)$$

$X = Spec(\mathbb{Q})$ $\pi_1^{et}(X) = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ *[NT]*

X over $\mathbb{C}$ $\pi_1^{et}(X) = \widehat{\pi_1}^{top}(X(\mathbb{C})^{an}, x)$ loops *[Geo]*

Arithmetic & Geometry – GGA **[Insight]**

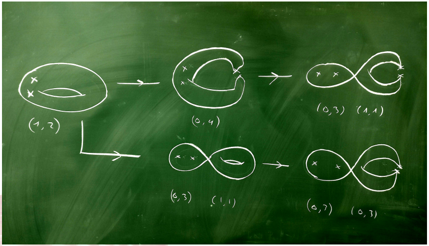
Fibration $X_* \longrightarrow X \longrightarrow \mathrm{Spec}\,\mathbb{Q}$

$$1 \to \underset{Geo-cont.}{\pi_1^{et}(X \otimes \bar{\mathbb{Q}}, *)} \to \pi_1^{et}(X,*) \to \underset{NT-disc.}{Gal(\bar{\mathbb{Q}}/\mathbb{Q})} \to 1$$

$\rightsquigarrow$ Geometric Galois action

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to Out[\widehat{\pi_1}^{top}(X(\mathbb{C})^{an}, x)]$$

Families of curves $\mathcal{M}_{g,[m]}$ - Thurston stratification (Esquisse) **[Encounter]**



$\Rightarrow$ **Description of GGAs for all (g,m)?**

- *In Algebraic Geometry: Fam. of curves*

$$\bar{\mathcal{M}}_{g_1,m_1} \times \bar{\mathcal{M}}_{g_2,m_2} \to \bar{\mathcal{M}}_{g_1+g_2,m_1+m_2-1}$$

- *Grothendieck-Teichmüller theory:* `Ihara, Matsumoto, Nakamura, Lochak, Schneps`

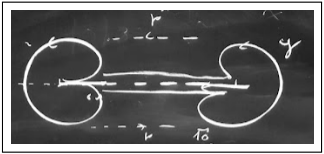
"There exists a GGA on every $\mathcal{M}_{0,[m]}$, combinatorially defined on braid groups, given by the dim. 1 & 2, i.e. by the GGAs on (0,4) and (0,5) only"

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{arith.} Out[\widehat{\pi_1}^{top}(\mathcal{M}_{0,[m]}(\mathbb{C})^{an}, x)]$$

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \hookrightarrow \widehat{GT} \xrightarrow{geo.} Out[\widehat{\pi_1}^{top}(\mathcal{M}_{0,[m]}(\mathbb{C})^{an}, x)]$$

$$\widehat{GT} = \{(\chi, f) \in \widehat{\mathbb{Z}}^{\times} \times \widehat{\mathbb{F}}_2 \text{ with Eq. I, II, III}\}$$

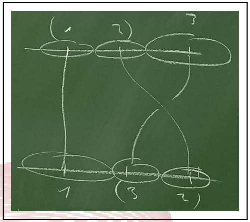
Via *Galois analytic transport* $x \rightsquigarrow y$



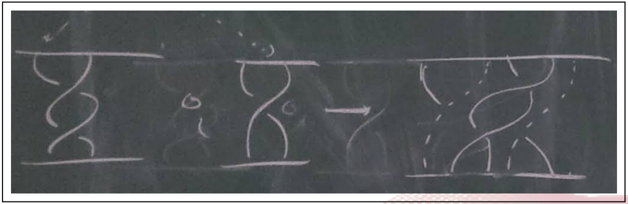
On $\mathcal{M}_{0,4} \simeq \mathbb{P}^1 \setminus \{0,1,\infty\}$: $(\chi, f) \in \widehat{\mathbb{Z}} \times \widehat{\mathbb{F}}_2$

Quillen Model Cat. & Operads. $\pi_1^{et}(X) \rightsquigarrow (X; fib, cofib., weq.)$ **[Encounter]**

The Operad of little 2-discs $E_2 = \{E_2(n)\}_{n\geq 0}$: $\pi E_2 \simeq PaB$ *Parenthesized Braid Op.*



Morphism in PaB(3)



Composition $PaB(2) \times PaB(2) \to PaB(3)$

$\Rightarrow$ **A geometric result:** $\widehat{GT} \simeq hAut(E_2^{\wedge})$ (Fresse, Horel)

Moduli problem & spaces classification [Encounter]

The category fibred in groupoids $\mathcal{M}\colon \mathcal{M}_{g,[m]} \to \mathrm{Sch}_{\mathbb{Q}}$ with for $S \in \mathrm{Sch}_{\mathbb{Q}}$ $\mathcal{M}_{g,[m]}(S)$ families of curves $C \to S$ of genus g with m marked points.

- Is a Deligne-Mumford stack over $\mathbb{Q}$
- Is a *fine* moduli for curves *with* automorphisms (Groupoids)

Stack inertia and automorphisms. Curves isomorphism $C_x \overset{\alpha}{\simeq} C_y$ are 2-morphisms

$$\mathrm{Spec}\,k \underset{y}{\overset{x}{\rightrightarrows}} \mathcal{M}_{g,[m]} \quad (\alpha\colon x \Rightarrow y)$$

via base-change

$$\begin{array}{ccccc} \alpha \in I_{\mathcal{M},x} & \dashrightarrow & I_{\mathcal{M}} & \dashrightarrow & \mathcal{M} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Spec}\,k & \xrightarrow{x} & \mathcal{M} & \longrightarrow & \mathcal{M} \times \mathcal{M} \end{array}$$

The $I_{\mathcal{M},x}$ are *finite group* schemes *orthogonal* to the classical GGAs.

Stack Arithmetic & GT [Insight]

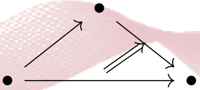
There exists a (cyclic) stack inertia Galois action

$$I^{cyc}_{\mathcal{M},\bar{x}} \hookrightarrow \pi_1^{et}(\mathcal{M}_{g,[m]} \otimes \bar{\mathbb{Q}}, *) \curvearrowleft Gal(\bar{\mathbb{Q}}/\mathbb{Q})$$

that (1) is $\widehat{GT}$-compatible and (2) provides a definition of $\widehat{GT}$.

Remarques. [Cat & Disc.-Cont.]

1. *Disc.-Cont.:* The atlas $\mathcal{M}_{g,[m]} \leftarrow X$ "rigidifies" $\mathcal{M}$
2. *Mod. Cat.:* $\mathcal{M} \in Sh(\mathrm{Sch}_{\mathbb{Q}}, Gpds) \leftrightarrows sPr(\mathrm{Sch}_{\mathbb{Q}})$
   Étale homotopy type $\pi_1^{et}(X) \rightsquigarrow \{X\}_{et} \in \text{Prop-Sp} \rightsquigarrow \pi_N^{et}(-)$



The category of Mixed Tate motives $MT(\mathbb{Z}) = \langle \mathbb{Z}(i) \rangle_i$ is a Tannaka category with *motivic Galois grp* G_{MT} $(g = 0)$.

$$\begin{array}{ccc} G_{MT} & \longrightarrow & Out[\widehat{\pi}_1^{\,uni}(\mathbb{P}^1 \setminus \{0,1,\infty\})] \\ & \searrow & \uparrow \\ & & GT(\mathbb{Q}) = \mathbb{G}_m(\mathbb{Q}) \ltimes GT^u(\mathbb{Q}) \end{array}$$

The category of Mixed Motives for $\mathrm{Sm}_{\mathbb{Q}} \xrightarrow{h} \mathrm{MM}$ is inspired by topology:

- $\mathbb{A}^1$-invariance: $h(X) \simeq h(X \times \mathbb{A}^1)$
- Mayer-Vietoris: $X = U \cup V \rightsquigarrow \ldots \to h(U \cap V) \to h(U) \oplus h(V) \to h(X) \xrightarrow{+1} \ldots$

Algebraic Topology & Spectra **[Encounter]**

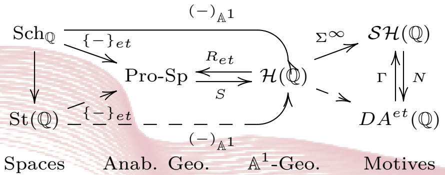
Topological spectra: $\mathcal{E} = \{E_i\}_{i \geq 1}$ seq. of spaces with bonding maps $\Sigma \wedge E_n \to E_{n+1}$

- *Harer Stability:* $H_k(\mathcal{M}_{g,r}, \mathbb{Q}) \xrightarrow{\sim} H_k(\mathcal{M}_{g+1,r-2}, \mathbb{Q})$ for $g \geq 3k - 1$, $r \geq 2$
- *Brown rep.:* $\mathcal{F} \colon Top \to grAb$ with cohomology axioms is represented by a spectrum... ($\rightsquigarrow$ stable homotopy/homology groups of $\mathcal{E}$).

$\Rightarrow$ ***Stable motivic homotopy category*** (Morel-Voevodski)

$$\mathcal{SH}(\mathbb{Q}) = Ho[Sp(sPr_{Nis}(\mathrm{Sm}_{\mathbb{Q}}); \mathbb{A}^1\text{-eq.}] \text{ for } \Sigma = (\mathbb{P}^1, \infty)$$

Anabelian & Motivic Geometries - Unifying... **[Insight]**



$\rightsquigarrow$ *Stack Inertia as a topological loop space:* $I_{\mathcal{M}} = [\mathbb{S}^1, \mathcal{M}]$

$\rightsquigarrow$ *In* $(\mathcal{S})\mathcal{H}(\mathbb{Q})$ *a decomposition* $\mathbb{P}^1 \simeq \mathbb{G}_m \wedge \mathbb{S}^1$

From $\pi_1^{et}(X)$, how/can we recover $X \in Sch_k$? *(Fukugen)*
$\rightsquigarrow$ Initiated by `Nakamura` & `Tamagawa` over $k/\mathbb{Q}$ number field.

Tate-Faltings p-adic Hodge theory **[Encounter]**

For $K/\mathbb{Q}_p$ p-adic field, comparison isom.

$$(\pi_1^{(p)}(X \otimes \bar{K})^{ab} \otimes_{\mathbb{Z}_p} \mathbb{C}_p)^{G_K} \simeq \underbrace{H^0(X, \omega_X)}_{\Gamma_X^\omega}$$

Container: $X \hookrightarrow \mathbb{P}(\Gamma_X^\omega)$
Points: $X \ni x_\infty = \lim_{L/K \in Fet.} x_L$

Mono-anabelian rec. & Transport "There exists a gr theoretic algo..." **[Insight]**

- *For* $\pi_1 = G_K$*:* Recover grps $(\bar{K}^\times, \boxtimes)$ and $(\bar{K}, \boxplus)$... but not the field $(\bar{K}, \boxtimes, \boxplus)$.
  $\Rightarrow$ *Beyond ring-scheme theory.*
- *Anabelian transport* between Étale & Frobenius obj. via cyclotome sync.:

$$^{\dagger}\bar{K} \xrightarrow{\kappa} H^1(G_{^{\dagger}K}, \Lambda(^{\dagger}\bar{K})) \underset{\sim}{\xrightarrow{\{\pm 1\}-Ind}} H^1(G_{^{\ddagger}\bar{K}}, \Lambda(^{\ddagger}\bar{K})) \xrightarrow{\kappa^{-1}} {}^{\ddagger}\bar{K}$$

Unifying Anabelian & Diophantine Geometries
$\rightsquigarrow$ Globalize & Sync. various $p \in \mathbb{Z}$: **categories of Hodge Theatres** $\mathcal{HT}$
$\Rightarrow$ Height inequalities for Elliptic curves: abc, Fermat, sections conj.:
Inter-universal Teichmüller theory

Anabelian GT
From anab. geo. to GGAs:
$\Rightarrow \widehat{GT} \geqslant BGT \rightsquigarrow \bar{\mathbb{Q}}_{BGT}$

A Combinatorial description of $\bar{\mathbb{Q}}$

Thank you for your attention